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On the motion of a micropolar fluid drop in a viscous fluid

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SUMMARY

The problems of the flow of a viscous fluid past a micropolar fluid sphere and the flow of a micropolar fluid past a viscous fluid drop are discussed. The expressions for the stream functions, velocities, spins and the drag are obtained in each case and are compared with the classical (viscous fluid past a viscous fluid sphere) results. It is found that the viscosity ratios and the parameter s, which arises in connection with the boundary condition, have significant effect upon the drag on the sphere in each case.

1. Introduction

In the recent years considerable attention has been given to fluid mechanical theories in which the couple stress, in addition to the traditional Cauchy stress, and the spin of the fluid particle, in addition to the usual velocity vector, play a significant role. These fluids have been suggested to describe the complex behaviour of such materials as liquid crystals, fluid suspensions and the blood flow. Eringen [1] has introduced a theory for such fluids, which are called by him micropolar fluids, in which the fluid can support stress and body couples and possess a rotation field which is independent of the velocity field. The theory, thus, has two independent kinematical variables: the velocity vector v_i and the spin or microrotation vector σ_i. The linear constitutive equation for the stress contains an additional material coefficient, which describes the coupling between v_i and σ_i. Also the linear equation for the couple stress contains three additional viscosity coefficients.

In the present paper we consider the two related problems of the flow of a viscous fluid past a fluid sphere which has a micropolar fluid inside it and the flow of a micropolar fluid flow past a viscous fluid sphere. It is assumed that the fluid spheres, in each case, remain spherical permanently. This problem of the translation of a viscous fluid drop in another viscous fluid was first considered by Rybczynski [2] and Hadamard [3] and is treated by Happel and Brenner [4] in their monograph. We shall follow the notations of [4] as much as possible. Ramkissoon and Majumdar [5] have studied the flow of a micropolar fluid past a solid sphere while Avudainayagam [6] has obtained the effective viscosity of a dilute suspension of micropolar fluid drops in a viscous fluid without explicitly calculating velocity, pressure etc. In this paper we determine velocity, spin, drag etc., in both the cases and compare them with the classical values. It is found that, as expected, the viscosity ratio and the parameter s, which appears in a compromise boundary condition, relating the spin of the particle with the vorticity of the fluid, have significant effects in the physical quantities of interest.

2. Basic equations

Neglecting the thermal effects and assuming the fluid to be incompressible with isotropic microstructure, the equations of continuity and the momenta, for a micropolar fluid, are given as [1]:

$$v_{i,i} = 0, \tag{1}$$

$$(\kappa + \mu)v_{i,jj} + \kappa\epsilon_{ijk}\sigma_{k,j} - p_{,i} + \rho f_i = \rho\dot{v}_i, \tag{2}$$

$$(\alpha + \beta)\sigma_{j,ij} + \gamma\sigma_{i,jj} + \kappa\epsilon_{ijk}v_{k,j} - 2\kappa\sigma_i + \rho C_i = \rho I\dot{\sigma}_i. \tag{3}$$

Here v_i is the velocity, σ_i the spin vector, ϵ_{ijk} the permutation tensor, p the pressure, f_i the body force, C_i the body couple, ρ the mass density and I is the local micro-inertia. The quantities μ, κ, α, β and γ are viscosity coefficients and are all assumed to be constant. Also a dot signifies material differentiation and the comma denotes partial differentiation with respect to a space coordinate.

The constitutive equation for the stress tensor $t_{k\ell}$ and the couple stress tensor $m_{k\ell}$ are given as

$$t_{k\ell} = -p\delta_{k\ell} + \mu(v_{k,\ell} + v_{\ell,k}) + \kappa(v_{\ell,k} - \epsilon_{k\ell m}\sigma_m), \tag{4}$$

$$m_{k\ell} = \alpha\sigma_{p,p}\delta_{k\ell} + \beta\sigma_{k,\ell} + \gamma\sigma_{\ell,k}. \tag{5}$$

Furthermore the requirement that the energy dissipation be nonnegative implies that:

$$2\mu + \kappa \geqslant 0, \quad \kappa \geqslant 0, \quad 3\alpha + \beta + \gamma \geqslant 0, \quad \gamma \geqslant 0, \quad \gamma \geqslant |\beta|. \tag{6}$$

3. Micropolar fluid drop in a viscous fluid

We shall first consider the problem of a micropolar fluid sphere in the steady flow of a viscous fluid. We assume that the fluid sphere is at rest while the viscous fluid streams past it with uniform velocity U.

It is convenient to work with the spherical polar coordinates (r,θ,ϕ) with the origin being the drop centre and with the $\theta = 0$ axis being taken in the direction of the free stream flow. Since the fluid sphere is assumed to maintain the spherical shape permanently, the flow both inside as well as outside of the sphere can be assumed axisymmetrical. Accordingly for the motion inside the drop we take

$$v_i = (v_r, v_\theta 0), \quad \sigma_i = (0,0,\sigma_\phi). \tag{7}$$

For the motion of the viscous fluid outside the drop we shall adopt the known solution [4] with the appropriate boundary conditions. Following [4] we introduce the stream function ψ such that

$$v_r = -\frac{1}{r^2 \sin\theta}\frac{\partial\psi}{\partial\theta}, \quad v_\theta = \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}. \tag{8}$$

When equations (7) and (8) are employed in equations (2) and (3) respectively and the inertia terms are neglected, we obtain (cf. [5])

$$E^4(E^2 - \delta^2)\psi = 0, \tag{9}$$

$$2r\sin\theta\,\sigma_\Phi = \left(E^2 + \frac{\gamma(\mu+\kappa)}{\kappa^2}E^4\right)\psi, \tag{10}$$

where δ is of dimension $(\text{length})^{-1}$, defined as

$$\delta^2 = \kappa(2\mu+\kappa)/\gamma(\mu+\kappa), \tag{11}$$

and

$$E^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right). \tag{12}$$

In order to solve (9), we note that, since the operators are commutative, we can write

$$\psi = \psi_1 + \psi_2 \tag{13}$$

where

$$E^4\psi_1 = 0 \tag{14}$$

and

$$(E^2 - \delta^2)\psi_2 = 0. \tag{15}$$

The solution for ψ_1, suitable for the present purpose (cf. [4]), is given as

$$\psi_1 = \left(\frac{A_1}{r} + B_1 r + C_1 r^2 + D_1 r^4\right)\sin^2\theta \tag{16}$$

In order to solve (15) we write $\psi_2 = f(r)\sin^2\theta$ and make a change of variable $f = \sqrt{\nu}\,g$ where $\nu = \delta r$, to obtain

$$\nu^2 g'' + \nu g' - (\nu^2 + \tfrac{9}{4})g = 0. \tag{17}$$

Here the dashes denote differentiation with respect to ν. The solution of (17) is

$$g = A_2 I_{3/2}(\nu) + B_2 I_{-3/2}(\nu),$$

where $I(\nu)$ is the modified Bessel function of the first kind. Hence

$$\psi_2 = \sqrt{\delta r}\,[A_2 I_{3/2}(\delta r) + B_2 I_{-3/2}(\delta r)]\sin^2\theta. \tag{18}$$

However, since

$$I_{3/2}(\nu) = \sqrt{\frac{2}{\pi\nu}}\left(\cosh\nu - \frac{\sinh\nu}{\nu}\right),$$

$$I_{-3/2}(\nu) = \sqrt{\frac{2}{\pi\nu}}\left(\sinh\nu - \frac{\cosh\nu}{\nu}\right),$$

we can rewrite (18) in one of the following forms:

$$\psi_2 = \sqrt{\frac{2}{\pi}}\left[\left(A_2 - \frac{B_2}{\delta r}\right)\cosh\delta r + \left(B_2 - \frac{A_2}{\delta r}\right)\sinh\delta r\right]\sin^2\theta \tag{19}$$

or

$$\psi_2 = \left[C_2\left(1 - \frac{1}{\delta r}\right)e^{\delta r} + D_2\left(1 + \frac{1}{\delta r}\right)e^{-\delta r}\right]\sin^2\theta. \tag{20}$$

Thus on combining (16) and (20) we have the solution for the stream function ψ as:

$$\psi = \left[\left\{\frac{\delta A_1 - (C_2 - D_2)}{\delta r}\right\} + \left\{\frac{2!B_1 + \delta(C_2 - D_2)}{2!}\right\} r + \right.$$

$$\left. + \left\{\frac{3!C_1 + 2\delta^2(C_2 + D_2)}{3!}\right\} r^2 + D_1 r^4 + \sum_{n=3}^{\infty} \frac{(C_2 + (-1)^n D_2) n\delta^n r^n}{(n+1)!}\right]\sin^2\theta. \tag{21}$$

The requirement that the velocities remain finite at the origin implies that

$$\delta A_1 = C_2 - D_2, \quad B_1 = -\frac{\delta^2}{2} A_1. \tag{22}$$

Also the non-zero component of the spin, σ_Φ, then reduces to

$$\sigma_\Phi = \left[\frac{(2\mu + \kappa)}{\gamma}\left\{\delta A_1\left(-\frac{1}{\delta r^2} + \sum_0^\infty \frac{(2m+1)\delta^{2m+1} r^{2m}}{(2m+2)!}\right)\right.\right.$$

$$\left.\left. + G_1 \sum_1^\infty \frac{2m\delta^{2m} r^{2m-1}}{(2m+1)!}\right\} + 5D_1 r\right]\sin\theta, \tag{23}$$

where $G_1 = C_2 + D_2$. Moreover the requirement that σ_Φ should remain finite at $r = 0$ implies that

$$A_1 = B_1 = 0, \quad C_2 = D_2. \tag{24}$$

Thus the stream function, the velocity components, the spin component and the tangential stress for the micropolar fluid inside the fluid sphere are given as:

$$\psi = \left[C_1 r^2 + D_1 r^4 + 2C_2 \left(\cosh(\delta r) - \frac{\sinh(\delta r)}{\delta r} \right) \right] \sin^2\theta,$$

$$v_r = -2 \left[C_1 + D_1 r^2 + 2C_2 \left(\frac{\cosh(\delta r)}{r^2} - \frac{\sinh(\delta r)}{r^3} \right) \right] \cos\theta,$$

$$v_\theta = 2 \left[C_1 + 2D_1 r^2 + C_2 \left(\frac{-\cosh(\delta r)}{r^2} + \frac{(1+\delta^2 r^2)}{\delta r^3} \sinh\delta r \right) \right] \sin\theta, \quad (25)$$

$$\sigma_\Phi = \left[5D_1 r + \frac{2(\mu+\kappa)}{\kappa} \delta^2 C_2 \left(\frac{\cosh(\delta r)}{r} - \frac{\sinh(\delta r)}{\delta r^2} \right] \sin\theta,$$

$$t_{r\theta} = (2\mu+\kappa)\sin\theta \left[3D_1 r + 2C_2 \left\{ \frac{3}{r^3} \cosh(\delta r) - \frac{(\delta^2 r^2 + 3)}{\delta r^2} \sinh(\delta r) \right\} \right].$$

For the slow motion of the viscous fluid outside the sphere, we adopt the known solution as [4]:

$$\overline{\psi} = \left(\frac{A_0}{r} + B_0 r + C_0 r^2 + D_0 r^4 \right) \sin^2\theta, \quad (26)$$

which on the requirement that $\overline{\psi} \to \frac{U}{2} r^2 \sin^2\theta$ as $r \to \infty$ reduces to

$$\overline{\psi} = \left[\frac{A_0}{r} + B_0 r + \frac{U}{2} r^2 \right] \sin^2\theta. \quad (27)$$

Moreover, we also obtain

$$\overline{v}_r = -\left[\frac{2A_0}{r^3} + \frac{2B_0}{r} + U \right] \cos\theta,$$

$$\overline{v}_\theta = \left[-\frac{A_0}{r^3} + \frac{B_0}{r} + U \right] \sin\theta, \quad (28)$$

$$\overline{t}_{r\theta} = \frac{6\overline{\mu} A_0}{r^4} \sin\theta.$$

Here, we have used bars to denote the quantities for the viscous fluid outside the sphere.

We now proceed to determine the remaining arbitrary constants C_1, D_1, C_2, A_0 and B_0 by employing the following boundary conditions in equations (25) and (28):

(i) The fluid is impenetrable at the surface of the sphere $r = a$, i.e. $v_r = 0$, and $\overline{v}_r = 0$ at $r = a$.

(ii) The tangential velocity is continuous at the interface, i.e. $v_\theta = \overline{v}_\theta$ at $r = a$.

(iii) The tangential stress is continuous across the boundary $r = a$, i.e. $t_{r\theta}|_{r=a} = \overline{t}_{r\theta}|_{r=a}$.

(iv) The spin vorticity relation at the interface is given by $\sigma_\phi|_{r=a} = s\overline{\omega}_\phi|_{r=a}$, $0 \leq s \leq 1$, where $\overline{\omega}_\phi$ is the vorticity component of the viscous fluid outside the drop.

As a result of the application of these conditions, we obtain

$$
\begin{aligned}
& C_1 + a^2 D_1 + 2\left(\frac{\delta a \cosh(\delta a) - \sinh(\delta a)}{\delta a^3}\right) C_2 = 0, \\
& 2A_0 + 2a^2 B_0 = -Ua^3, \\
& -\frac{A_0}{a^3} + \frac{B_0}{a} - 2C_1 - 4a^2 D_1 + \\
& + 2\left\{\frac{\delta a \cosh(\delta a) - (1+\delta^2 a^2)\sinh(\delta a)}{\delta a^3}\right\} C_2 = -U, \\
& 6\overline{\mu} A_0 - 3(2\mu+\kappa)a^5 D_1 - \\
& -2(2\mu+\kappa)\left\{\frac{3\delta a \cosh(\delta a) - (3+\delta^2 a^2)\sinh(\delta a)}{\delta}\right\} C_2 = -U, \\
& 5a^3 D_1 + 2\delta(\mu+\kappa)\{\delta a \cosh(\delta a) - \sinh(\delta a)\} C_2 = -sB_0 .
\end{aligned} \tag{29}
$$

The solution of the system of equations (29) is given as:

$$
\begin{aligned}
A_0 &= \frac{L}{2(1+\lambda_1)} + \frac{5}{2(1+\lambda_1)}\left\{(M-N) - \frac{\delta^2 a^2}{3} N\right\}\overline{C}_2, \\
B_0 &= -\frac{(3+2\lambda_1)}{2(1+\lambda_1)}\frac{L}{a^2} - \frac{5}{2(1+\lambda_1)}\left\{(M-N) - \frac{\delta^2 a^2}{3} N\right\}\frac{\overline{C}_2}{a^2}, \\
C_1 &= \frac{-\lambda_1}{2(1+\lambda_1)}\frac{L}{a^3} - \frac{1}{2(1+\lambda_1)}\left\{5\lambda_1(M-N) - \frac{(3\lambda_1-2)}{3}\delta^2 a^2 N\right\}\frac{\overline{C}_2}{a^3}, \\
D_1 &= \frac{\lambda_1}{2(1+\lambda_1)}\frac{L}{a^5} + \frac{(3\lambda_1-2)}{2(1+\lambda_1)}\left\{(M-N) - \frac{\delta^2 a^2}{3} N\right\}\frac{\overline{C}_2}{a^5},
\end{aligned} \tag{30}
$$

where

$$
L = \frac{Ua^3}{2}, \quad M = a\cosh(\delta a), \quad N = \frac{\sinh(\delta a)}{\delta}, \quad \lambda_1 = \frac{2\overline{\mu}}{(2\mu+\kappa)}, \tag{31}
$$

$$
\overline{C}_2 = 2C_2 = \frac{3\kappa[3s + 2(s-5)\lambda_1]L}{5\kappa[3\lambda_1 - (2+s)]\{3(M-N) - \delta^2 a^2 N\} + [6(1+\lambda_1)(\mu+\kappa)(M-N)\delta^2 a^2]} .
$$

We point out that the first term on the right hand side, for each of A_0, B_0, C_1 and D_1, is essentially the same as in the case of the drop of a viscous fluid (cf [4]). The other terms related to $\overline{C}_2$ are due to the presence of the spin in the micropolar fluid. As expected, these terms systematically disappear in the limit as κ and or γ tend to zero. Because of the axisymmetrical nature of the flow assumed and because of the similar dependence of the θ coordinate terms in both the classical (when the fluid inside the drop is viscous) and in the present case, the overall pattern of the stream lines appears similar in both the cases. In particular, the circulation within the droplet, as is observed in the classical case, is also predicted in the present situation. The difference in the present case is, however, that stream-lines are displaced towards or away from the origin depending upon the magnitudes of s and λ_1. Similar remarks apply for the stream lines outside of the sphere.

We next calculate the drag on the sphere by using the standard integral formula

$$\bar{D} = 2\pi a^2 \int_0^{\pi} (t_{rr}\cos\theta - t_{r\theta}\sin\theta)\,|_{r=a} \sin\theta\, d\theta. \tag{32}$$

Upon calculating t_{rr} and $t_{r\theta}$ with the use of equations (4), (25), (28) and (30) we finally obtain

$$\bar{D} = -6\pi\bar{\mu}aU\left[\frac{1+\frac{2}{3}\lambda_1}{1+\lambda_1}\right] - \frac{6\pi a\bar{\mu}U}{(1+\lambda_1)} \times$$

$$\times \left[\frac{\{s+\frac{2}{3}(s-5)\lambda_1\}\{3(M-N)-\delta^2a^2N\}}{\{3\lambda_1-(2+s)\{3(M-N)-\delta^2a^2N\}+\{\frac{6}{5}(\mu+k)(1+\lambda_1)/k\}}\right]. \tag{33}$$

If we denote by $\bar{D}_1$ the classicial drag (i.e. the drag in the case of a viscous fluid drop) as given in [4],

$$\bar{D}_1 = -6\pi\mu aU\frac{(1+\frac{2}{3}\lambda_1)}{(1+\lambda_1)},$$

then the above expression can be rewritten as

$$\frac{\bar{D}}{\bar{D}_1} \cong \frac{5k\left[\frac{(3\lambda_1+2)(2\lambda_1-3)}{(2\lambda_1+3)}\right][3(M-N)-\delta^2a^2N]+[6(1+\lambda_1)(\mu+\kappa)(M-N)\delta^2a^2]}{5k[3\lambda_1-(2+s)][3(M-N)-\delta^2a^2N]+[6(1+\lambda_1)(\mu+\kappa)(M-N)\delta^2a^2]} \tag{34}$$

From equation (34) we note that $\bar{D}/\bar{D}_1$, apart from the other quantities, depends significantly upon the values of λ_1 and s. It is seen that for large values of λ_1 i.e. when $\lambda_1 \geq .375$ and for all values of $s \in [0,1]$, $\bar{D}/\bar{D}_1 < 1$. However, when $\lambda_1 < .375$ there is a small region where $\bar{D}/\bar{D}_1 > 1$ is possible for different values of s. In particular we note that for values close to 1 for s and for smaller values of λ_1 $(.01 \leq 1 \leq .37)$, $\bar{D}/\bar{D}_1$ is greater than one. Thus we conclude that when the fluid inside the sphere is a micropolar fluid then the drag on the sphere is, in general, smaller as compared to when the droplet is filled with a viscous fluid. However, for certain values of s and λ_1, the drag on the sphere filled with the micropolar fluid could be larger than that in the case of the viscous fluid.

4. Viscous drop in a micropolar fluid

We shall now consider the problem when the fluid inside the drop is a viscous fluid while the fluid streaming past the sphere is a micropolar fluid. If, as before, we assume that the fluid sphere maintains the spherical shape permanently, then the general solutions obtained in the previous section can be adapted with the slight modifications of the appropriate boundary conditions. Thus for the motion of the viscous fluid inside the drop we again assume:

$$\psi^{(i)} = \left[\frac{A_4}{r} + B_4 r + C_4 r^2 + D_4 r^4\right] \sin^2\theta,$$

which, upon the requirement that velocities be finite at the centre of the droplet, reduces to

$$\begin{aligned}
\psi^{(i)} &= [C_4 r^2 + D_4 r^4] \sin^2\theta, \\
v_r^{(i)} &= -2\cos\theta\, [D_4 r^2 + C_4], \\
v_\theta^{(i)} &= \sin\theta\, [4D_4 r^2 + 2C_4], \\
t_{r\theta}^{(i)} &= \sin\theta\, [6\mu^{(i)} D_4 r].
\end{aligned} \tag{35}$$

Similarly for the slow motion of the micropolar fluid outside the sphere we write the expression for the stream function, using equations (16) and (20), as

$$\psi^{(0)} = \left[\frac{A_3}{r} + B_3 r + C_3 r^2 + D_3 r^4 + E_3\left(1 - \frac{1}{\delta' r}\right) e^{r\delta'} + \right.$$

$$\left. + F_3\left(1 + \frac{1}{r\delta'}\right) e^{-r\delta'}\right] \sin^2\theta. \tag{36}$$

Since $\psi^{(0)} \to \frac{1}{2} U r^2 \sin^2\theta$ as $r \to \infty$, we find $D_3 = E_3 = 0$ and $C_3 = \frac{1}{2} U$. Hence $\psi^{(0)}$ and $\sigma_\phi^{(0)}$ now reduce to

$$\psi^{(0)} = \left[\frac{A_3}{r} + B_3 r + \frac{U}{2} r^2 + F_3\left(1 + \frac{1}{r\delta'}\right) e^{-r\delta'}\right] \sin^2\theta, \tag{37}$$

$$\sigma_\phi^{(0)} = \left[-\frac{B_3}{r^2} + F_3\left(\frac{\mu' + \kappa'}{\kappa'}\right) \delta' \left\{\frac{1 + r\delta'}{r^2}\right\} e^{-r\delta'}\right] \sin\theta. \tag{38}$$

The other components of interests can now be written as

$$\begin{aligned}
v_r^{(0)} &= -2\cos\theta \left[\frac{A_3}{r^3} + \frac{B_3}{r} + \frac{U}{2} + F_3\left\{\frac{1 + r\delta'}{r^3\delta'}\right\} e^{-r\delta'}\right], \\
v_\theta^{(0)} &= \sin\theta \left[-\frac{A_3}{r^3} + \frac{B_3}{r} + U - F_3\left\{\frac{3 + r\delta' + r^2\delta'}{r^4\delta'}\right\} e^{-r\delta'}\right], \\
t_{r\theta}^{(0)} &= 2(\mu' + \kappa') \sin\theta \left[\frac{3A_3}{r^4} + F_3\left\{\frac{r^2\delta'^2 + 3\delta' r' + 3}{r^4\delta'}\right\} e^{-r\delta'}\right].
\end{aligned} \tag{39}$$

We may point out that we have used the superscripts i and 0, respectively, for the quantities inside and outside of the drop. Also the dashes have been added to the material coefficients of the micropolar fluid in order to distinguish them from the material coefficients of the previous section. The first three boundary conditions, (i)-(iii), employed in the previous section are applicable in the present case also. These lead to the following set of four simultaneous equations:

$$C_4 + a^2 D_4 = 0,$$

$$A_3 + a^2 B_3 + F_3 \left(\frac{1 + a\delta'}{\delta'} e^{-a\delta'} \right) = -\frac{U}{2} a^3,$$

$$-A_3 + a^2 B_3 - F_3 \left(\frac{1 + a\delta' + a^2\delta'^2}{\delta'} \right) e^{-a\delta'} = -Ua^3 + 2C_4 a^3 + 4a^3 D_4, \tag{40}$$

$$3A_3 + F_3 \left\{ \frac{\delta'^2 a^2 + 3\delta' a + 3}{\delta'} \right\} e^{-a\delta'} = 6 \frac{\mu^{(i)} a D_4}{2(\mu' + k')}.$$

The solution of (40) is

$$A_3 = \frac{L}{2(1+\lambda_2)} - \left\{ 1 + \delta' + \frac{3 + 2\lambda_2}{6(1+\lambda_2)} a^2 \delta'^2 \right\} G_3,$$

$$B_3 = \frac{-(3 + 2\lambda_2)}{2(1+\lambda_2)} \frac{L}{a^2} + \frac{3 + 2\lambda_2}{6(1+\lambda_2)} \delta'^2 G_3,$$

$$C_4 = \frac{-\lambda_2}{2(1+\lambda_2)} \frac{L}{a^3} + \frac{\lambda_2}{6(1+\lambda_2)} \frac{\delta'^2}{a} G_3, \tag{41}$$

$$D_4 = \frac{\lambda_2}{2(1+\lambda_2)} \frac{L}{a^5} - \frac{\lambda_2}{6(1+\lambda_2)} \frac{\delta'^2}{a^3} G_3,$$

where

$$L = \frac{Ua^3}{2}, \quad \lambda_2 = \frac{2\mu' + \kappa'}{2\mu^{(i)}}, \quad G_3 = \frac{F_3}{\delta' e^{a\delta'}}, \quad \delta'^2 = \frac{\kappa'(2\mu' + \kappa')}{\gamma'(\mu' + \kappa')}. \tag{42}$$

The final boundary condition, involving the spin-vorticity relation, is taken as

$$\sigma_\phi^{(0)}|_{r=a} = s\omega_\phi^{(i)}|_{r=a}, \quad 0 \leq s \leq 1,$$

On using (35) and (38) the above condition reduces to

$$-B_3 + \frac{\delta'^2(\mu' + \kappa')}{\kappa'} (1 + a\delta')G_3 = 5a^3 s D_4 \tag{43}$$

which, upon the substitution of the values of D_4 and B_3 from equation (41) gives

$$G_3 = \left[\frac{3\kappa'\{3 + (2 - 5s)\lambda_2\}}{\{3 + (2 - 5s)\lambda_2\}\kappa' - \{6(1 + \lambda_2)(\mu' + \kappa')(1 + a\delta')\}} \right] \frac{L}{a^2 \delta'^2}. \tag{44}$$

It can be observed, from equations (35), (37), (41) and (44) that, in the present case also, the general features of the stream-lines, both inside and outside the drop, are similar to those observed in the classical case. The presence of the micropolar fluid outside the drop changes the stream-lines inside the drop slightly. However, circulation inside the droplet is predicted in the present case also. We point out that when $\kappa \to 0$, the above results reduce to the classical case [4], while when $\lambda_2 \to 0$ (i.e. when the sphere is considered solid) the above expressions reduce to those given by Ramkissoon and Majumdar [5].

The calculation of the drag on the fluid sphere can be carried out in exactly the same manner as in the previous section. After using formula (32) and simplifying we get

$$\bar{D}_2 = -6\pi\left(\mu' + \frac{\kappa'}{2}\right) aU\left[\frac{1+\frac{2}{3}\lambda_2}{1+\lambda_2}\right] \times$$
$$\times\left[\frac{6(1+\lambda_2)(\mu'+\kappa')(1+a\delta')}{6(1+\lambda_2)(\mu'+\kappa')(1+a\delta') - \kappa'\{3+(2-5s)\lambda_2\}}\right]. \tag{45}$$

If, as before, we define the classical drag by $\bar{D}_1$, we then have

$$\frac{\bar{D}_2}{\bar{D}_1} \simeq \left[\frac{6(1+\lambda_2)(\mu'+\kappa')(1+a\delta')}{6(1+\lambda_2)(\mu'+\kappa')(1+a\delta') - \kappa'\{3+(2-5s)\lambda_2\}}\right]. \tag{46}$$

From equation (46) we observe that $\bar{D}_2/\bar{D}_1 > 1$ when s is small i.e. when $s \leq .5$, for all values of λ_2. However, for values of s closer to one and $\lambda_2 > 1$ there is a region where $\bar{D}_2/\bar{D}_1 < 1$. Moreover for $\lambda_2 < 1$ and for all values of s we again find $\bar{D}_2/\bar{D}_1 > 1$. Thus for higher values of λ_2, if s is closer to one we have $\bar{D}_2/\bar{D}_1 < 1$ while if s is closer to zero we have $\bar{D}_2/\bar{D}_1 > 1$. We, therefore, conclude that the drag on a viscous sphere moving in a micropolar fluid depends strongly upon the magnitude of the viscosity ratio λ_2, and the parameter s. Depending upon the values of s and λ_2, it could be greater or lesser as compared to the classical drag. This result is slightly different from the result of the drag on a solid sphere moving in a micropolar fluid in which case the drag is found to be always greater than the classical drag (cf. [5]).

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